Evaluation of Optimal Control Strategies

JOHN E. COTTER
TRW Computers Company, Canoga Park, California

In a recent article* in this journal, a computational method for minimal time control of nonlinear systems was presented. The generality of the method was based on the claim that Silva and LaSalle had shown that optimum control is on the boundaries of its limits except possibly for the very end of the transient response. This is not a correct interpretation. Silva required this to be the control action and demonstrated that the response was close to optimal for high-order linear systems. LaSalle also treated linear systems. Therefore, no general conclusions may be drawn for nonlinear systems. As a matter of fact, time-optimal control of a particular nonlinear process may never be on the boundaries. A good example may be found in reference *10* of the article by Grethlein and Lapidus.

One useful method for examining optimal strategies is the maximum principle of Pontriagin (*1*). If one is given a system described by n equations in the state variables (x_i) and control variables (u_j)

$$\frac{dx_i}{dt} = f_i(x_i, u_j, t)$$

and a performance function to be minimized

$$\int_0^\tau f_o\,(x_i, u_j, t)\;dt$$

then the maximum principle states that the Hamiltonian function

$$H\,(x_i, \chi_i, u_j, t) = \sum_{i=0}^{i} \chi_i f_i$$

must necessarily be a maximum along the trajectory for $0 \leqq t \leqq \tau$. The adjoint variables, χ_i, are defined as

$$\frac{d\chi_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad i = 1 \ldots n$$

χ_o is constant
and $(\chi_1, \chi_2 \ldots \chi_n) \neq 0$

An interesting comparison may be seen in the minimum time operation of a batch reactor with a single exothermic reaction controlled by a cooling coil. The reaction is pseudo-first order.

Irreversible reaction: rate $= -x\,k_1(T)$ where x = reactant mole fraction, T = reaction temperature, $k_1(T)$ = specific reaction velocity.

The system equations are then

$$\frac{dx}{dt} = -\frac{x\,V\,k_1(T)}{W_x}$$

$$\frac{dT}{dt} = \frac{VH_r x\,k_1\,(T)}{W_T} - [T - T_o]\,\frac{\varphi}{W_T}$$

Now $f_o = 1$ for minimum time control, so that

$$H = \chi_o - \chi_1\,x\,\frac{V}{W_x}\,k_1(T) + \frac{\chi_2}{W_T}\,\{VH_r x\,k_1(T) - [T - T_o]\varphi\}$$

The control strategy is determined by the coefficient of φ, $\frac{-\chi_2}{W_T}\,[T - T_o]$.

In order to maximize H

$$\varphi = \min \varphi \text{ if } \chi_2 > 0$$
$$\varphi = \max \varphi \text{ if } \chi_2 < 0$$

This is bang-bang control; but if χ_2 goes to zero for any finite time, more investigation is required. This implies that χ_2 and $\frac{d\chi_2}{dt}$ must go to zero together. But

$$\frac{d\chi_2}{dt} = -\frac{\partial H}{\partial T} = \chi_1\,x\,\frac{V}{W_x}\,\frac{dk_1}{dT} - \frac{\chi_2}{W_T}\,[VH_r x\,\frac{dk_1}{dT} - \varphi]$$

so χ_1 would have to go to zero with χ_2; but this is impossible, since the vector

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* Grethlein, H. E., and Leon Lapidus, *A.I.Ch.E. Journal*, 9, No. 2, p. 230 (1963).

Comments on the Above Communication to the Editor

LEON LAPIDUS
Princeton University, Princeton, New Jersey

The basic premise of the paper by Grethlein and Lapidus is to ascertain an optimal control sequence for a nonlinear process subject to the constraints of physical reality. In many physical systems the input is automatically of the bang-bang type; this paper then illustrates the implications and results for one approach to handling this problem. It may be that in certain cases the controlled results are not truly optimal but the approach to optimality is probably quite close and within the limits imposed by a partial knowledge of the process model.

The minor amount of discussion in the paper dealing with the theoretical justification for bang-bang minimum time control was, of course, only to provide a broad-gauged foundation for its use. Unfortunately this discussion should have specified that only normal-type systems were under consideration. Thus if the system model is given by the linear equations (all in vector-matrix form)

$$\dot{x}(t) = A\,x(t) + B\,m(t)$$

where A and B are constant matrices then the solution is given by

$$x(t) = X(t)\,x(0) + X(t)\int_0^t X^{-1}(\lambda)\,B\,m(\lambda)\,d\lambda$$

where $X(t)$ satisfies

$$\dot{X}(t) = A\,X(t) \quad X(0) = I$$

For every fixed $t \geqq 0$ there is a subset $C(t)$ of n-dimensional space which represents the set of all initial conditions from which the origin may be reached in t time units. If the actual initial condition x_0 is given then LaSalle has shown that there is at least one new vector η such that

$$-\eta \cdot x_0 \geqq \eta \cdot w \text{ for all } w \in C(t_o)$$

This means that the function $\eta \cdot w$ takes on its maximum in $C(t_o)$ when $w = -x_0$.

The system under consideration is said to be normal if no component of $[\eta \cdot X^{-1}(t)\,B]$ vanishes on any interval, no matter what the vector $\eta \neq 0$. In this case the optimal control is bang-bang, and unique, and is given by

$$m(t) = \text{sgn}\,[\eta \cdot X^{-1}(t)B]$$

Obviously this analysis applies only to linear systems and merely indicates a direction in the case of nonlinear systems. In the latter case, as Dr. Cotter has pointed out, the maximum principle may be used to define the type of optimal control. However, when there are more than a few state variables it is almost impossible to use this method to evaluate the explicit control strategy. Thus considerable research work must be done before a truly optimal control sequence for nonlinear systems may be always defined. The purpose of the paper under discussion was to illustrate one method of handling this problem.